Curriculum Vitae et Studiorum

Dario Mazzoleni

February 2017

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Current position

PostDoc at the University of Torino since December 1st, 2014, for the ERC grant "Complex patterns for strongly interacting dynamical systems" (https://compaterc.wordpress.com/), principal investigator: Prof. Susanna Terracini.

Studies

Ph.D. in Mathematics at the University of Pavia with a cotutelle agreement with the Friedrich-Alexander Universität of Erlangen-Nürnberg, obtained on December 2nd, 2014 in Pavia.
Title of the Thesis: Existence and regularity results for solutions of spectral problems
Advisors: Prof. Aldo Pratelli and Prof. Giuseppe Savaré

Members of the Committee:
President: Prof. Giuseppe Buttazzo (University of Pisa)
Prof. Aldo Pratelli (University of Erlangen)
Prof. Giuseppe Savaré (University of Pavia)
Prof. Frank Duzaar (University of Erlangen)

Referees:
Prof. Dorin Bucur (University of Savoy)
Prof. Giuseppe Buttazzo (University of Pisa)

Prof. Aldo Pratelli (University of Erlangen)

Previous studies

- Diploma of the **IUSS** (Institute of advanced studies of Pavia, `www.iusspavia.it`) obtained on January 31st, 2012 with mark: Excellent. Title of the thesis: *Minimizing movements and spatial discretization for functionals with free discontinuities*, advisor: Prof. Enrico Vitali, co-advisor: Prof. Franco Brezzi.
- *Laurea Magistrale* (**Master's degree**) in Mathematics, achieved on July 12th, 2011 at University of Pavia with grade 110/110 cum laude. Title of the Thesis: *Extremum problems, quantitative estimates and connectedness of optimal sets for spectral inequalities*, advisor: Prof. Aldo Pratelli.
- Fellow of *Almo Collegio Borromeo* (Pavia, `www.collegioborromeo.it`) during the years 2006-2011.
- *Laurea Triennale* (**Bachelor's degree**) in Mathematics, achieved on September 22nd, 2009 at University of Pavia with grade 110/110 cum laude. Title of the Thesis: *Stime del gradiente nei problemi con discontinuità libera* [Gradient bounds in free-discontinuity problems], advisor: Prof. Enrico Vitali.
- High school degree (scientific oriented) achieved in July 2006 at *Liceo Scientifico* "Filippo Lussana" of Bergamo (Italy), with grade 100/100.

Prizes and awards

- The paper (M1) in collaboration with Aldo Pratelli was awarded as a **Highly cited research** by the editors of J. Math. Pures Appl.
- Prize "Proff. Silvio Cinquini e Maria Cinquini Cibrario" for the best master's degree thesis in Mathematics at the University of Pavia in the period from 01.09.2010 to 31.08.2012.
- **Best student award** from the Faculty of Mathematics, Physics and Natural Sciences of the University of Pavia for the academic year 2010/11.
- IUSS study prize in the years 2006/07, 2007/08, 2008/09, 2009/10, 2010/11.

Other scientific activities

1. *Local organizer* of the Workshop "New trends in shape optimization", held in Erlangen on September 23-27, 2013.
2. **Referee** for "Bullettin of the London Mathematical Society" and "JEMS". **Reviewer** for MathSciNet since February 2016.
3. Since February 2015 I got the "Qualification aux fonctions de maître de conférences" for teaching in France in the section of Mathematics (25) and Applied Mathematics (26).
4. Since January 2015, member of the research group GNAMPA of INDAM (group of calculus of variations in the italian institute of high mathematics). Moreover I am a member of the project **OptiFrac**, p.i. Giovanni Franzina, March 2016–February 2017.

Talks

1. "XXII Convegno nazionale di Calcolo delle Variazioni" (Levico Terme, February 5-10, 2012): *Existence of minimizers for spectral problems in* $\mathbb{R}^N$.
2. "Workshop on Calculus of Variation and Partial Differential Equations" (Erlangen, November 9th, 2012): *Existence of minimizers for spectral problems in* $\mathbb{R}^N$.
3. Workshop "New trends in shape optimization" (Erlangen, September 23-27, 2013): *Existence and regularity for spectral problems.*
4. "XXIV Convegno nazionale di Calcolo delle Variazioni" (Levico Terme, January 27-31, 2014): *A regularity result for spectral problems.*
5. "School on free discontinuity problems" (Pisa, July 7-11, 2014): *A regularity result for spectral problems.*
6. Chambéry, September 12th, 2014, Seminar at the LAMA, invited by Dorin Bucur: *A regularity result for spectral problems.*
7. Berlin, March 6th, 2015, Seminar at WIAS, invited by Enrico Valdinoci: *Existence, regularity and surgery results for solutions of spectral problems.*
8. Lyon, April 7th, 2015, Seminar at the Université Lyon 1, invited by Louis Dupaigne: *Surgery and regularity results for spectral problems.*
9. "Lions-Magenes days" (Pavia, April 13-14, 2015): *A surgery result for the spectrum of the Dirichlet Laplacian.*
10. Workshop "Shape optimization and spectral geometry" (Edinburgh, June 29th- July 3rd, 2015): *A surgery result for the spectrum of the Dirichlet Laplacian.*
11. XX Congresso U.M.I. (Siena, September 7-12th, 2015): *Surgery results for spectral problems.*
12. Workshop "Calculus of variations and PDEs" (Chambéry, September 24-25th, 2015): *Geometric properties of optimal sets for some spectral optimization problems.*
13. Trieste, January 12th, 2016, Seminar at the SISSA, invited by Davide Zucco: *Surgery results for spectral problems.*
14. Workshop "Analytical and Geometric properties of solutions of PDEs" (Napoli, January 25-27th, 2016): *Geometric properties of optimal sets for some spectral optimization problems.*
15. Marseille, March 31st, 2016, Seminar at the FRUMAM - Université Aix-Marseille, invited by Enea Parini: *Existence et régularité pour des problems spectraux.*
16. Workshop "Geometric aspects of PDE's and functional inequalities" (Cortona, April 28-30th, 2016): *Geometric properties of optimal sets for some spectral optimization problems.*
17. Workshop "Bruxelles-Torino talks in PDE's" (Torino, May 2-5th, 2016): *Geometric properties of optimal sets for some spectral optimization problems.*
18. Workshop "Geometric and analytic inequalities" (BIRS, Banff–Canada, July 10-15th, 2016): *Regularity of optimal sets for spectral functionals.*
19. Workshop "GMT, shape optimisation and free boundaries" (SISSA Trieste, October 25th-28th, 2016): *Regularity of the optimal sets for some spectral functionals.*
20. Workshop "Shape Optimization and Isoperimetric and Functional Inequalities" (CIRM Marseille, November 21st-25th, 2016): *Regularity of the optimal sets for spectral functionals. Part II, some generalizations.*
21. Brescia, December 7th, 2016, Seminar at the Department of Maths and Physics of Università Cattolica, invited by Marco Squassina: *Regularity of the optimal sets for spectral functionals.*
22. Bruxelles, February 14th, 2017, Seminar at the Departément de Mathématique of ULB, invited by Denis Bonheure: *Regularity of the optimal sets for spectral functionals.*

Schools and workshops attended (without giving a talk)

- *ERC Summer School on Calculus of Variation, Continuum Mechanics and Quantitative inequalities* in Ischia, 5-10 June 2011. Organizers: A. Pratelli and F. Maggi.

- Workshop in memory of Enrico Magenes *Analysis and Numerics of Partial Differential Equations* in Pavia, 2-4 November 2011.
- Workshop *Variational Methods for Evolution* at MFO (Mathematisches Forschungsinstitut Oberwolfach), 5-9 December 2011. [Organizers: A. Mielke, F. Otto, G. Savaré, U. Stefanelli]
- Workshop *Trends in Mathematical Analysis* in Milan-Politecnico, 1-3 March 2012.
- Workshop *Shape optimization and spectral problems* at CIRM (Marseille), 28 May-1 June 2012. [Organizers: D. Bucur, G. Buttazzo, A. Henrot, M. Pierre]
- School on *Shape optimization* at Centro De Giorgi (Pisa), 4-8 June 2012. [Courses held by D. Bucur, B. Kawohl, M. Pierre]
- Workshops *Geometric inequalities in Calculus of Variation* and *New trends in Shape Optimization* at Centro De Giorgi (Pisa) 9-20 July 2012. [Organizers: N. Fusco & A. Pratelli, A. Henrot]
- *XXIII Convegno nazionale di Calcolo delle Variazioni*, Levico Terme, 4-8 February, 2013.
- *Seventh summer school in analysis and applied mathematics*, Rome, 17-21 June, 2013. [Courses held by F. Otto, G. Savaré & L. Truskinowsky]
- Workshop *Variational views in mechanics and materials*, Pavia, 24-26 June, 2013. [Organizers:G. Dal Maso et al.]
- *Vector-valued partial differential equations and applications*, Cetrato (CS), 8-12 July, 2013. [Courses held by: B. Dacorogna, N. Fusco, S. Müller & V. Sverak]
- *ERC school "Geometric functional inequalities and shape optimization"*, Napoli, 9-13 September 2013. [Courses held by: D. Bucur, F. Maggi & M. Morini]
- Workshop *"Calculus of Variations and Optimization. A conference to celebrate the 60th birthday of Giuseppe Buttazzo"*, Pisa, 21-23 May 2014. [Organizers: L. Brasco, A. Briani, G. Dal Maso, L. De Pascale, I. Fragalà, F. Santambrogio, P. Trebeschi]
- Workshop *Isoperimetric problems between analysis and geometry*, Pisa, 16-20 June 2014. [Organizers: G. De Philippis, A. Pratelli, F. Morgan]
- Workshop *Complex Patterns in nonlinear phenomena*, Torino, 26-30 January 2015.
- *XXV Convegno nazionale di Calcolo delle Variazioni*, Levico Terme, 2-6 February, 2015.
- Winter school/workshop *Spectral theory and shape optimization problems for elliptic PDEs*, Milano, 9-13 February 2015.
- Summer school *Geometric measure theory and calculus of variations*, Grenoble, June 22-26, 2015
- Workshop "Calculus of variations and PDEs" (Chambéry, September 24-25th, 2015).
- *XXVI Convegno nazionale di Calcolo delle Variazioni*, Levico Terme, January 18th-22nd, 2016.
- *Pde's at the Gran Paradis*, Cogne, June 20th-24th, 2016.
- *A mathematical tribute to Ennio de Giorgi*, Pisa, September 19th-23rd, 2016.
- *Roma Caput PDE*, Roma, January 23rd-26th, 2017.
- *XXVI Convegno nazionale di Calcolo delle Variazioni*, Levico Terme, February 6th-10th, 2017.

Teaching

At the University of Pavia.

1. Tutor for the course of Mathematics for first year students in Chemistry in the years 2009/10 and 2010/11. Lecturer: Prof. Enrico Vitali.
2. Teaching assistant for the course of Ordinary differential equations in the year 2011/12. Lecturer: Prof. Enrico Vitali.

3. Teaching assistant for the course of Sobolev spaces in the year 2011/12. Lecturer: Prof. Aldo Pratelli.

At the University of Torino

1. Responsible of the exercise group "Problem solving" (mainly exercises of functional analysis and measure theory) in the year 2014/15 and 2015/16.
2. Tutor for the course of Calculus I, for first year students in Mathematics, in the year 2015/16.
3. Ph.D. course (jointly taught with Eleonora Cinti): "An introduction to geometric measure theory", in the year 2016/17.

Spoken languages

Italian (mother tongue); English (fluent); French (basic); German (basic).

Long periods abroad

1. From November 1^{st}, 2012 to August 31^{st}, 2014 I was in Erlangen (Germany) as a Ph.D. student.
2. From September 1^{st}, 2014 to October 31^{st}, 2014 I was in Chambéry (France) as a guest of Dorin Bucur.

Computer skills

1. Good knowledge both of Windows and Linux (Ubuntu) systems, of the Office package and of LaTeX.
2. Basic knowledge of the languages "C" and "Matlab".

Other activities

1. Member of the school council of Liceo Scientifico "Filippo Lussana" during the school year 2005/06.
2. Representative of PostDocs at the council of the Department of Mathematics in Torino since September 2015.

Publications and preprints

One can find a preprint version of all my publications here: `http://cvgmt.sns.it/person/977/`.

M1) D. Mazzoleni, A. Pratelli, *Existence of minimizers for spectral problems*, J. Math. Pures Appl. **100** (3) 433-453 (2013).
This paper was awarded the certificate of **highly cited research**.

M2) M. Iversen, D. Mazzoleni, *Minimising convex combinations of low eigenvalues*, ESAIM:COCV **20** (2) 442–459 (2014).

M3) D. Bucur, D. Mazzoleni, A. Pratelli, B. Velichkov, *Lipschitz regularity of the eigenfunctions on optimal domains*, Arch. Ration. Mech. Anal. **216** (1) 117–151 (2015).

M4) D. Bucur, D. Mazzoleni, *A surgery result for the spectrum of the Dirichlet Laplacian*, SIAM J. Math. Anal. **47** (6) 4451–4466 (2015).

M5) D. Mazzoleni, *Recent existence results for spectral problems*, chapter of the book: "New trends in shape optimization", G. Leugering and A. Pratelli eds., Birkhäuser, 2015.

M6) D. Mazzoleni, *Boundedness of minimizers for spectral problems in* $\mathbb{R}^N$, Rend. Sem. Mat. Univ. Padova **135** (2016) 207–221.

M7) D. Mazzoleni, D. Zucco, *Convex combinations of low eigenvalues, Fraenkel asymmetries and attainable sets*, to appear on ESAIM:COCV, preprint available at `http://cvgmt.sns.it/person/977/`.

M8) D. Mazzoleni, S. Terracini, B. Velichkov, *Regularity of the optimal sets for some spectral functionals*, to appear on Geom. Funct. Anal., preprint available at `http://cvgmt.sns.it/person/977/`.

Papers in preparation

M9) D. Mazzoleni, A. Pratelli, *Some remarks on the eigenvalues of a set with* $\lambda_1(\Omega) \approx \lambda_1(B)$.

Past and current research

My main research interests are in the field of the Calculus of Variations and Shape Optimization. Many of my results deals with the following spectral minimization problem of which we investigate existence, regularity and properties of optimal sets:

$$\min\left\{F(\lambda_1(A),\dots,\lambda_k(A))\ :\ A\subset\mathbb{R}^N,\ |A|=1\right\}, \tag{1}$$

where we denote λ_i the i^{th} eigenvalue of the Dirichlet Laplacian, while $|\cdot|$ is the N-dimensional Lebesgue measure.

1) **Existence and surgery results.** For functionals F increasing in each variable and lower semicontinuous we proved with A. Pratelli in [22] the existence of a solution in the class of quasi-open sets, thus extending a previous result by Buttazzo and Dal Maso [11], which worked only in a bounded setting. Our main idea consists in showing that an open and regular set cannot have both "long tails" and the first k eigenvalues of the Dirichlet Laplacian "small". Hence, using a "surgery" argument, one can choose a minimizing sequence for problem (1) made of sets with bounded diameter, and then apply the result by Buttazzo and Dal Maso to this sequence.
Moreover it is also possible to prove, with similar arguments, that *all* minimizers have diameter uniformly bounded by a constant depending only on k, N [21], up to assume a slightly stronger hypothesis on F.
The "surgery" techniques developed in [22] for proving an existence result for problem (1) did not give any information on the perimeter of the rearranged set. With D. Bucur [9] we studied this issue and proved that, given an open set Ω with unit measure, it is possible to build a new open set of unit measure and bounded diameter with both the first k eigenvalues and the perimeter decreased. This argument allows

also to give an existence result for a shape optimization problem with a double constraint of both measure and perimeter, at least in the class of measurable sets (with a suitable new definition of eigenvalues and H_0^1 in this setting).

2) **Regularity results and free boundary problems.** A major open problem in spectral shape optimization consists in proving the regularity of optimal sets, that in principle are only quasi-open. We are able to give [10] a partial answer to the question above, focusing mostly on the regularity of the eigenfunctions. Together with Bucur, Pratelli and Velichkov we attacked this problem using techniques coming from the free boundary theory which were used for proving the Lipschitz regularity for the Dirichlet energy function by Briançon, Hayouni and Pierre [4]. A careful reformulation of those ideas together with the use of *shape supersolutions*[1] make possible to prove also that every optimal set Ω_k^* for λ_k admits a k^{th} eigenfunction which is Lipschitz continuous in $\mathbb{R}^N$. Moreover, for "increasingly bi-Lipschitz" functionals, having actually in mind $\lambda_1(\cdot)+\cdots+\lambda_k(\cdot)$, we are also able to prove that all the first k eigenfunctions on minimizers are Lipschitz continuous and then that optimal domains are actually open.

The above-mentioned result lead to a work, with S. Terracini and B. Velichkov, about a more careful study of regularity issue for the functional $\lambda_1(\cdot)+\cdots+\lambda_k(\cdot)$. We prove an analogous result to the theorems proved by Alt, Caffarelli [1] and Weiss [26] for the regularity of the free boundary for the Dirichlet energy, that is, showing that every optimal set Ω^* is at least $C^{1,\alpha}$ regular (actually it is possible to deduce also C^∞ with a careful bootstrap argument), up to a singular set of Hausdorff dimension less than $N - N^*$, where $N^* \in [5, 7]$ is the smallest dimension at which minimizing free boundaries admit singularities. It is conjectured that $N^* = 7$, but up to our knowledge this has not been proved, yet. The main difference is that we have a vectorial setting and so we have to deal with functions that do not have constant sign.

The main steps of our proof are the following:

i) We reformulate the shape optimization problem in a free-boundary analogue, where our interest is the study of local (quasi-)minimizer for

$$H^1_{loc}(\mathbb{R}^N, \mathbb{R}^k) \ni V \mapsto \int |\nabla V|^2 + \Lambda|\{|V| > 0\}|.$$

We call U a local minimizer for the above functional for sake of simplicity.

ii) We prove a nondegeneracy lemma for the vector $U = (u_1, \ldots, u_k)$ and then show that actually if U is nondegenerate, than u_1 is nondegenerate, too. Then it is possible to prove also some density estimates on optimal sets.

iii) We give some monotonicity formulas (similar to the ones of Weiss [26]) and make a blow-up, showing that the limits are local minimizer and $1-$homogeneous.

iv) We prove an optimality condition for the gradient of the vector U on the boundary, in a "viscosity sense", following some ideas from De Silva [13].

v) We show how it is possible to reduce the vectorial optimality condition to one depending only on u_1 by means of a *boundary Harnack principle* and then we apply the usual improved flatness results for one-phase free boundary problems.

[1] Roughly speaking a shape supersolution for a functional defined on shapes is a set which is optimal with respect to external perturbations.

3) **Geometric properties of optimal sets.** With M. Iversen, in [18], we study a more specific shape optimization problem. We consider convex combinations of the first three eigenvalues of the Dirichlet Laplacian and we give informations about connectedness of optimal sets, with a particular attention to the two-dimensional case. The basic idea is very simple: one studies the optimal disconnected configuration and then looks for a better connected competitor. In particular we are able to prove that for all $\alpha \in (0,1]$, every minimizer for the functional $\alpha\lambda_1(\cdot) + (1-\alpha)\lambda_2(\cdot)$ is connected.
Still dealing with the functional $\alpha\lambda_1(\cdot) + (1-\alpha)\lambda_2(\cdot)$, with D. Zucco, we have proved that there exists $\overline{\alpha} > 0$ such that each optimal set Ω_α for F_α, if $\alpha \leq \overline{\alpha}$, is non-convex. This is done by a careful study of the optimal convex set for the so called $2-$Fraenkel asymmetry, which is a commonly used quantity in the field of quantitative estimates, which find here their first application to these kind of problems.

4) **Some estimates on the Dirichlet eigenvalues of a set with $\lambda_1(\Omega) \approx \lambda_1(B)$.** It is well known that two sets of the same measure can have the same first eigenvalue, but all the others eigenvalues very different from each other, for example one can take Ω_1 as the disjoint union of two equal balls and Ω_2 the disjoint union of a ball of half measure with many smaller balls. On the other hands, if a set $\Omega \subset \mathbb{R}^N$ of unit measure is such that $\lambda_1(\Omega) \approx \lambda_1(B)$, it is reasonable to conjecture that also all the higher eigenvalues should be close. In a paper in preparation with A. Pratelli we study this problem, and our main results are the followings. Let $k, N \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ be an open set of unit measure. There exists a constant $\overline{\vartheta}(k,N) < 1$ such that, if $\lambda_1(\Omega) - \lambda_1(B) \leq \overline{\vartheta}$, then the following estimates hold.

1. For all $\eta > 0$ there is a constant $C(k, N, \overline{\vartheta}, \eta)$ such that for all $j \leq k$,
$$\lambda_j(B) - \lambda_j(\Omega) \leq C(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{2N}-\eta}\,.$$
2. There exists a constant $C(k, N, \overline{\vartheta})$ such that for all $j \leq k$,
$$\lambda_j(\Omega) - \lambda_j(B) \leq C(\lambda_1(\Omega) - \lambda_1(B))^{1/8}\,.$$

These estimates can be proved mostly with rather elementary perturbation techniques.

Main research projects

1) **A gradient flow approach for eigenvalues.** We consider the minimization problem
$$\min\big\{\lambda_k(\Omega)\ :\ \Omega \subset \mathbb{R}^N,\ |\Omega| = 1\big\}. \tag{2}$$
The result [22] provides the existence of a quasi-open optimal set Ω_k^* for (2). A very interesting problem, that requires new techniques to be solved, consists in studying the "evolution" of the sets Ω_k^* when $k \to \infty$. A conjecture, supported by numerical results by Oudet [23], Antunes and Freitas [2], could be that the sequence Ω_k^* should converge to the ball, as $k \to \infty$.
It is not clear how to attack directly this conjecture, so we can split it into some intermediate steps.

i) Prove that the bound on the diameter given by the surgery result [22] can be improved so that it does not depend on k, as it seems to be reasonable from the numerical simulations (by Oudet [23], Antunes and Freitas [2]). This could be done either with an improvement of the surgery techniques from [22] or with a more careful study of shape subsolutions for the torsion energy, following some ideas by Bucur [6].

ii) Study the evolution of shapes $\Omega(t)$ starting from a given domain Ω_0, under a suitable notion of *gradient flow*. In particular we are interested in the evolution of sets under the cost functional λ_k. This is still a rather difficult issue, because it is not very clear what is a good metric for the space of quasi-open sets. A first try could be the γ–convergence, which was the main tool in the existence result by Buttazzo and Dal Maso [11]. Unfortunately, this metric is not compact in the class of quasi-open sets and, up to now, it is only known that there exist generalized minimizing movements for λ_k, using the weaker distance $d_{char}(A, B) = |A\Delta B|$. This was done in an inspiring paper by Bucur, Buttazzo and Stefanelli [8].

iii) Starting from the evolution problem highlighted in point *ii*), an easier intermediate step could be to prove the Faber–Krahn inequality (i.e. the unit ball has the lowest first eigenvalue among sets of unit measure) using the minimizing movements for quasi-open sets. In particular, it is sufficient to prove that minimizing movements for (λ_1, d_{char}) converge to stationary points for the functional λ_1 and that the *only* stationary point for λ_1 is actually the ball. Those two steps seem easier to approach, but nonetheless completely non-trivial. The good side of this path is that it does not require to prove the convergence of minimizing movements, but it is enough to know that they exist.

iv) Another possible path for analyzing the evolution of shapes could be to study the minimizing movements with other metrics, and one should start with the Hausdorff distance, in which case it is needed to consider only convex domains.

2) **Regularity issue.** An important open problem is to prove the regularity of minimizers for $\lambda_k(\cdot)$ (i.e. for solutions of problem (2)), which is the natural continuation of the work [10] and of the study of the regularity for $\lambda_1(\cdot)+\cdots+\lambda_k(\cdot)$. The case of the k^{th} eigenvalue is a lot harder, and we do not even know, at the moment, whether an optimal set Ω_k^* is actually open. We believe that it is possible to prove $C^{1,\alpha}$ regularity of the boundary of Ω_k^* (and then C^∞ should follow with a bootstrap argument), outside a singular set of Hausdorff dimension less than $N - N^*$, as in the case of the sum of the first k eigenvalues. In order to prove this strong result, an idea can be to perform some approximation procedure, as the one done in [25], where it is used the functional $(\lambda_1^p + \cdots + \lambda_k^p)^{1/p}$, which for $p \to \infty$ approaches λ_k. This is not at all easy to carry on, since one needs to prove uniform estimates on the eigenfunctions of the optimal set for the approximated problem, which is rather difficult.
A first intermediate step can be to prove regularity for functionals like $\lambda_1(\cdot) + \lambda_k(\cdot)$, for which the approximation procedure could be easier, thanks to the fact that λ_1 is preserved in the limit. The main parts of the proof should be similar to those done with S. Terracini and B. Velichkov for the sum functional $\lambda_1(\cdot) + \cdots + \lambda_k(\cdot)$, but one must pay attention that we do not know that the eigenfunctions that do not correspond to an eigenvalue involved in the minimization are Lipschitz continuous. This could create additional problems when proving, for example, monotonicity formulas.

3) **Shape optimization with a double constraint.** We consider the following problem with a measure *and* a perimeter constraint:

$$\min\big\{F(\lambda_1(A), \dots, \lambda_k(A)) \ : \ A \subset \mathbb{R}^N,\ |A| \le 1,\ P(A) \le P\big\}, \tag{3}$$

for some constant $P > 0$. The existence of an optimal set for (3) in the class of measurable sets (after suitably defining eigenvalues in this environment) follows from a result by Bucur, Buttazzo and

Henrot [7], together with the surgery argument proved in [9, Theorem 5.1]. The next problem is to study the regularity of optimal sets and hopefully prove that they are better than only measurables. A possible way to do this consists in trying to repeat for this problem the regularity results done in the case of the sole perimeter constraint, proved in [14, 15]. This is not at all trivial because, when dealing with two constraints, it is not immediate to pass from a constrained minimization to a Lagrange multiplier. Moreover the definition of eigenvalues on measurable sets is somewhat delicate and it should be kept into careful consideration.

4) **Spectral shape optimization for other operators.** Another field of research that can be interesting and full of developments is to extend the results [11, 22] to other operators, first of all the s-Laplacian, which is becoming increasingly studied in the last years. A first point can be to extend a generalization of the Buttazzo and Dal Maso existence Theorem [11] to eigenvalues for the s-Laplacian. This can be done, the main difficulty being to pay a lot of attention to the spaces in which one needs to work: in fact one key point is to find a suitable notion for the space $H_0^s(\Omega)$ when Ω is not regular. Moreover the space H_0^s can be defined as a meaningful space only if $s \in (1/2, 1)$. This is being studied in collaboration with M. Marini and B. Ruffini. The next step, which seems more difficult, is to study the minimization problem in the whole $\mathbb{R}^N$, as it was done for the local case in [22]. The structure of the fractional Laplacian leads to conjecture that there is a threshold $\overline{s} \in (1/2, 1)$ such that if $s \leq \overline{s}$ there is no minimum, but one can find an infimizing sequence made by at least two connected components going at infinity. This is somehow reminiscent of the dichothomy case in the concentration-compactness principle. A non-existence result of this kind was proved by Brasco and Parini [5] for λ_2^s (which is easier, since also for the usual Laplacian the optimal set has two connected components) and we aim to extend it in the general case for λ_k^s. The starting point is the study of what happens for λ_3^s, where the competition between the repulsive term from the s-Laplacian and the attractive term from the structure of the eigenvalues become clear. We believe this to be really the most interesting case, since the optimal set for the usual Dirichlet Laplacian is knwon to be connected (at least in dimension $N = 2, 3$) and the situation is very different from that of λ_2.

Another class of operators for which there are many shape optimization problems that can be studied are polyharmonic operators, starting from the biharmonic Laplacian. For the first eigenvalue of the biharmonic Laplacian it is known a Faber–Krahn like result in dimension $N = 2, 3$. The optimal set was proved to be the ball by Nadirashvili first and then by Ashbaugh and Benguria. It could be interesting to improve this result in a quantitative way. This means that, if a set is close to optimality, then it must be "close" to a ball, in particular its Fraenkel asymmetry should be controlled by the difference of the first eigenvalues to some exponent. In particular, the proof of Ashbaugh and Benguria is based on some rearrangements idea by Talenti, and it seems a suitable starting point for a quantitative improvement.

5) **Properties of minimizers of principal eigenvalue with indefinite weight and Neumann boundary conditions.** Given an open, connected and bounded set Ω, we consider the following eigenvalue problem with indefinite weight and Neumann boundary conditions:

$$\begin{cases} -\Delta u = \lambda m u & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $m \in L^\infty(\Omega)$ is a weight which changes sign in Ω. Given $k, m_0 > 0$, the existence of an optimal weight for the problem,

$$\min_{m \in \mathcal{M}} \lambda(m), \qquad \mathcal{M} := \left\{ m \in L^\infty(\Omega) \ : \ -1 \leq m \leq \kappa, \ m^+ \not\equiv 0, \ \int_\Omega m \leq -m_0|\Omega| \right\}, \tag{5}$$

was proved in the works [17] and [12], where in particular it is shown, using a bathtub principle, that the optimal m^* is of bang-bang type, i.e. $m^*(x) = \kappa\chi_D(c) - \chi_{\Omega\setminus D}(x)$ for some set $D \subset \Omega$. Moreover D is equal almost everywhere to a superlevel set of the first eigenfunction $\{u_1 > t\}$ for some $t > 0$. This optimal design problem can hence be rewritten as a shape optimization problem with D as unknown. This is what we aim to study with B. Pellacci and G. Verzini:

$$\inf_{D\in\mathcal{E}} \lambda(D), \qquad \mathcal{E} = \left\{ D \subset \Omega \ : \ \textit{measurable},\ 0 < |D| < \frac{1-m_0}{\kappa+1}|\Omega| \right\}. \tag{6}$$

This topic has been studied since many years and the literature is very wide. One of the most challenging problem was the study of the shape of an optimal D when the measure m_0 varies if the set $\Omega \subset \mathbb{R}^N$ is an $N-$rectangle. It was conjectured in [3, 16], following also some numerical results, that for values of m_0 close to zero, the optimal set D should have been a part of a ball. This longstanding conjecture was recently disproved in [20], where it is shown a more general result: if the set D is rotationally symmetric, then Ω must be a ball (or an annulus). Then, using a relation between Neumann and periodic boundary conditions, it is possible to see that it disproves the conjecture. The second part of the conjecture dealt with the case when $|D|/|\Omega \setminus D|$ is close to one and it stated that is this case the optimal D should be a strip. This is still open up to our knowledge in the general case (although sometimes in the literature it is given for granted, but without a proper proof), while there are some partial results [17, 19]. Concerning the non-local case when the operator is the spectral fractional Neumann Laplacian, there is a very recent result [24] where the basic existence facts for the problem are proved and also an optimization with respect to the fractional parameter $s \in (0, 1)$ is studied. On the other hand, nothing is known about the shape of the optimal set D, up to our knowledge. The first problem that we want to deal with consists in proving that, if $\Omega \subset \mathbb{R}^2$ is a rectangle, then the optimal set D must be a strip, when the ratio $|D|/|\Omega \setminus D|$ is close to one. The long term goal is to carry on an analysis of the optimal set D (at least when Ω is a rectangle) for the analogous problem for the spectral fractional Neumann Laplacian instead of the classical one. Finally we want also to investigate the more general case of Robin boundary conditions.

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